$\left(\text{Here, } x = t^3 - \frac{3}{2}t^2 \ ; \ y = t^3 + 3t^2 . \right)$

1. Let C be the parametrized curve given by

$$c(t) = (t^2(t - \frac{3}{2}), t^3 + 3t^2), \quad -\infty < t < \infty$$

(a) Find the points on the curve with vertical or horizontal tangent lines.

• $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2+6t}{3t^2-3t} = \dfrac{3t(t+2)}{3t(t-1)}$. (You can cancel the $3t$) (In fact, you should).

need to check!

Horizontal: $\dfrac{dy}{dx}$ must limit to zero. Candidates: When $3t(t+2) = 0$; $t = 0, -2$.

$t=0$ gives $\frac{0}{0}$; $\lim_{t\to 0} \dfrac{3t(t-2)}{3t(t-1)} \overset{\text{L'Hop}}{=} \lim_{t \to 0} \dfrac{6t-6}{6t-3} = 2 \neq 0$. So, $t=0$ is not.

-14

Only horizontally tangent at $t=-2 \to (x,y) = (-\cancel{20}, 4)$.

Vertical: $\left|\dfrac{dy}{dx}\right|$ must limit to ∞. Candidates: When $3t(t-1) = 0$; $t = 0, 1$.

We saw $\lim_{t\to 0} \dfrac{dy}{dx} = 2 \neq 0$ nor ∞, so it's not.

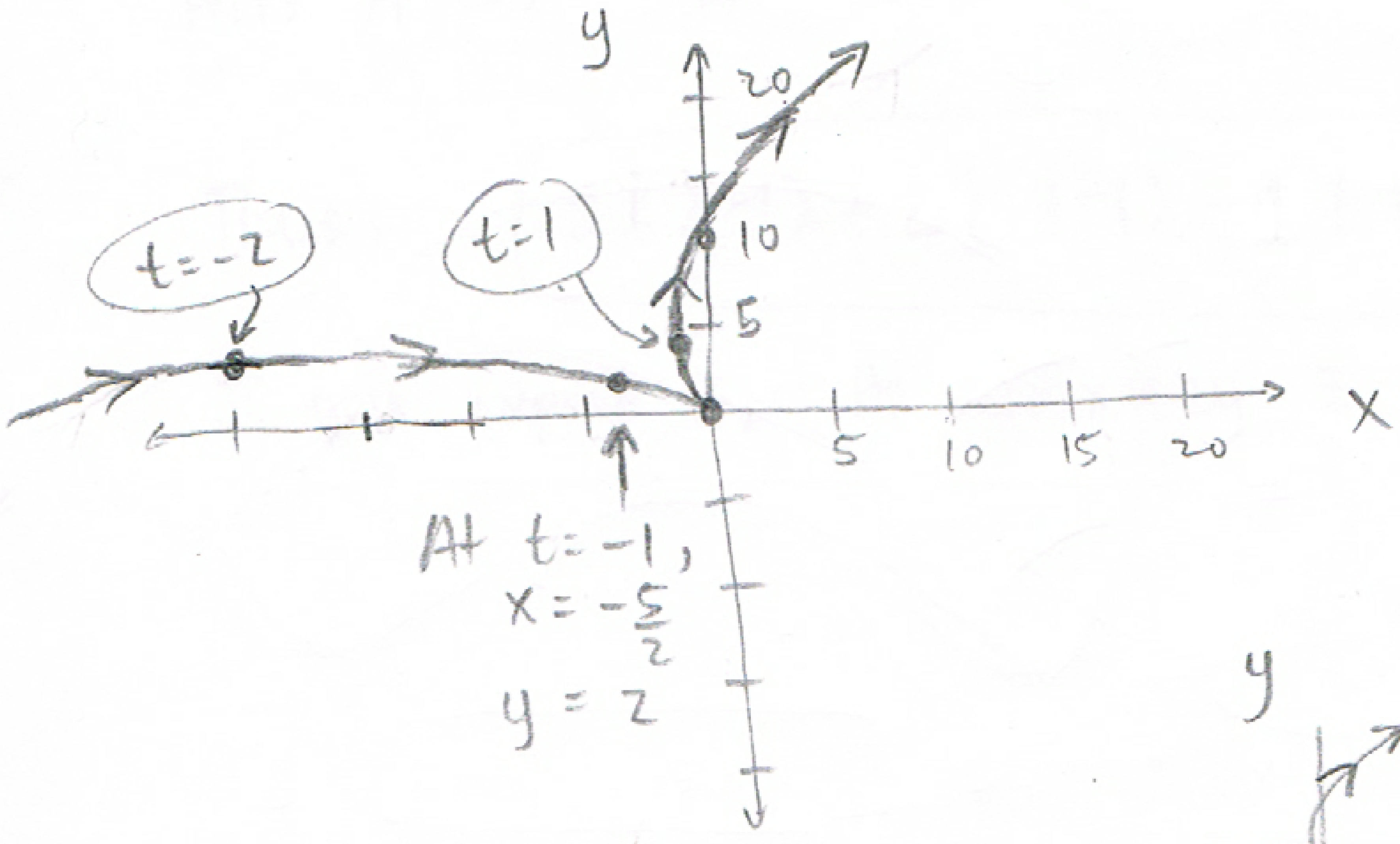
Only vertical tangent is at $t=1 \to (x,y) = (-\frac{1}{2}, 4)$.

(b) Sketch C roughly, by finding its x and y intercepts and by studying $\lim_{t\to\infty} c(t)$ and $\lim_{t\to-\infty} c(t)$. Use arrows to indicate the direction of the curve. Hint: Part (a) should also help to get a better picture of the curve.

is $\frac{81}{8} \approx 10$

$(x=0)$ • y-intercepts: $\boxed{t = 0, \frac{3}{2}} \to (0,0)$ and $(0, \frac{27}{8} + \frac{27}{4})$

$(y=0)$ • x-intercepts: $\boxed{t = 0, -3} \to (0,0)$ and $(-\frac{81}{2}, 0)$.



• When $t \to \infty$, x and y are both approximately $x \approx t^3$ and $y \approx t^3$

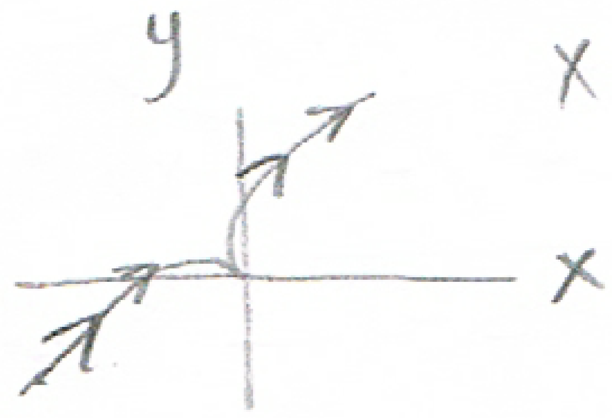
$\Rightarrow$ so its approx. $x=y$ line.

• Same when $t \to -\infty$.

☆ $y \geq 0$ on $t > -3$

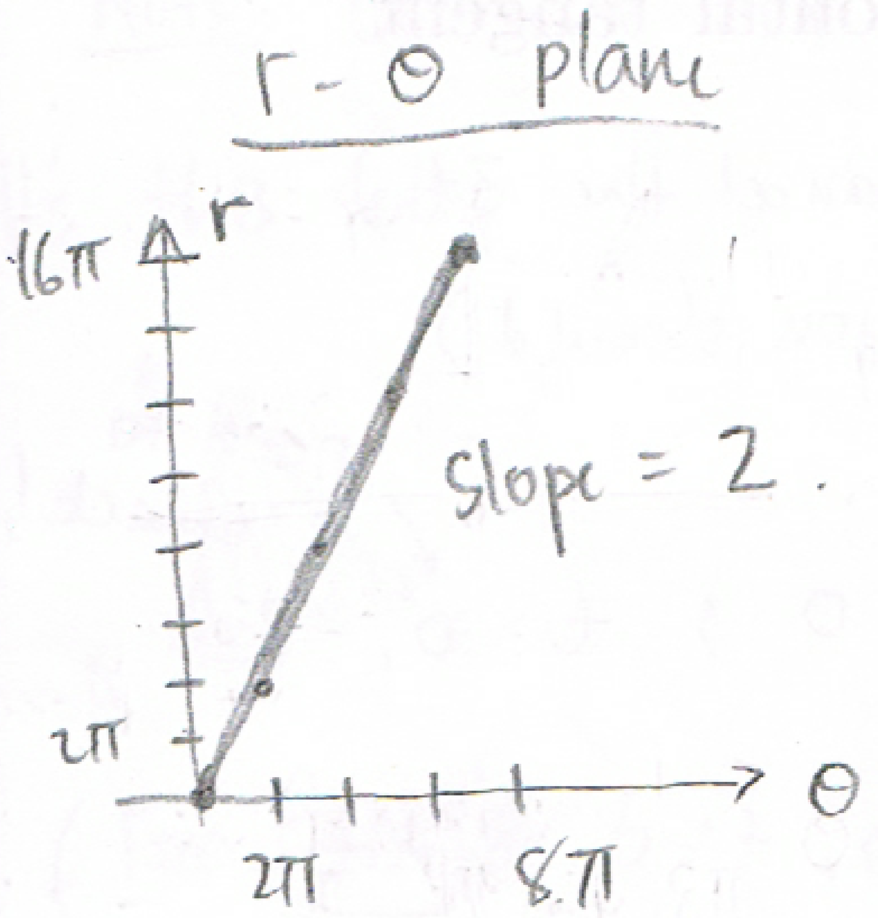
$x \geq 0$ on $t > 3/2$

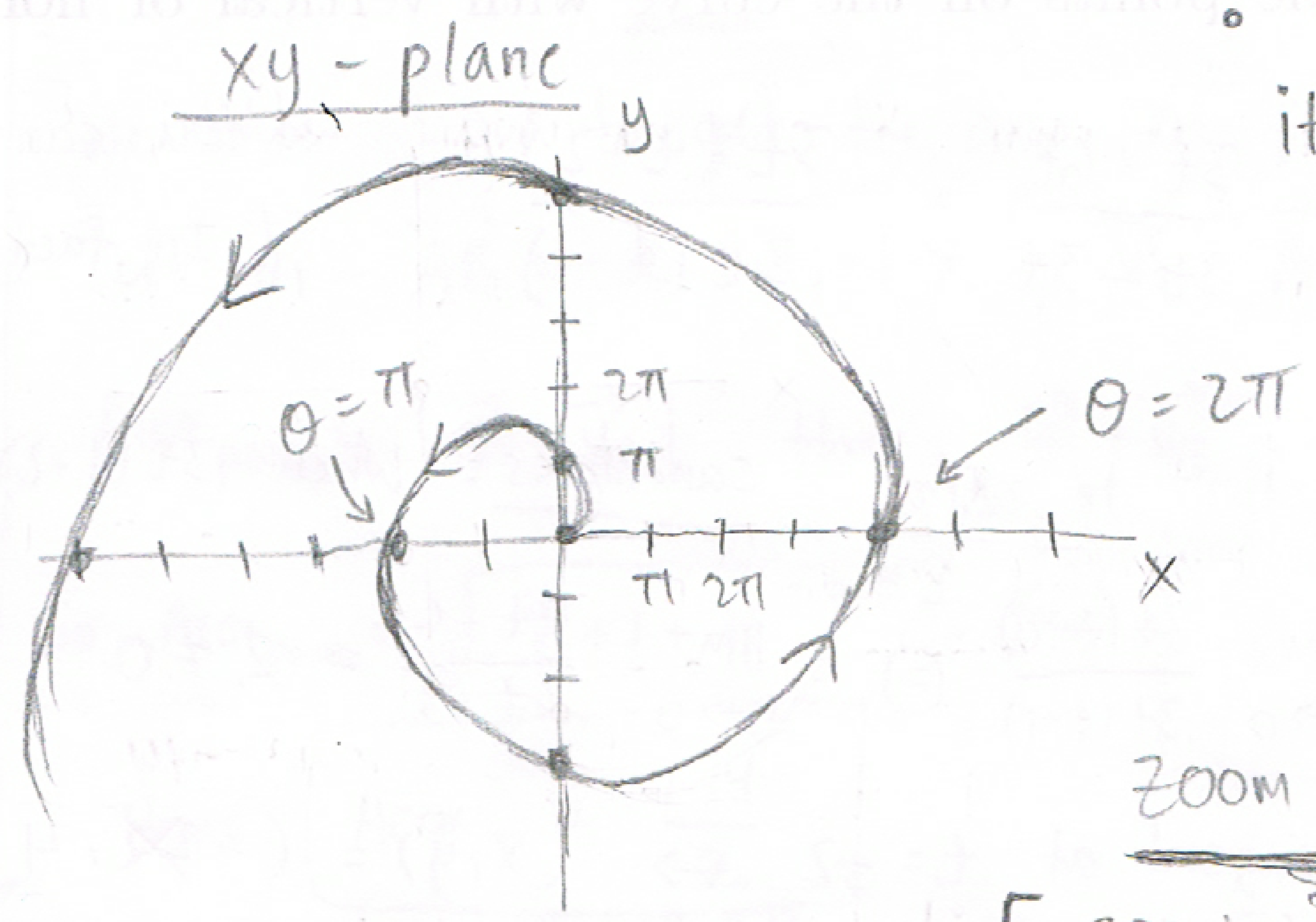
↳ zoom out:



2. Consider the polar curve $r = 2\theta$ for $0 \leq \theta \leq 8\pi$. Sketch this curve in the xy-plane and the $r\theta$-plane. Use arrows to show the direction of the curve. How many times does the curve rotate around the origin in the xy-plane.

(The 2θ is a trap!)



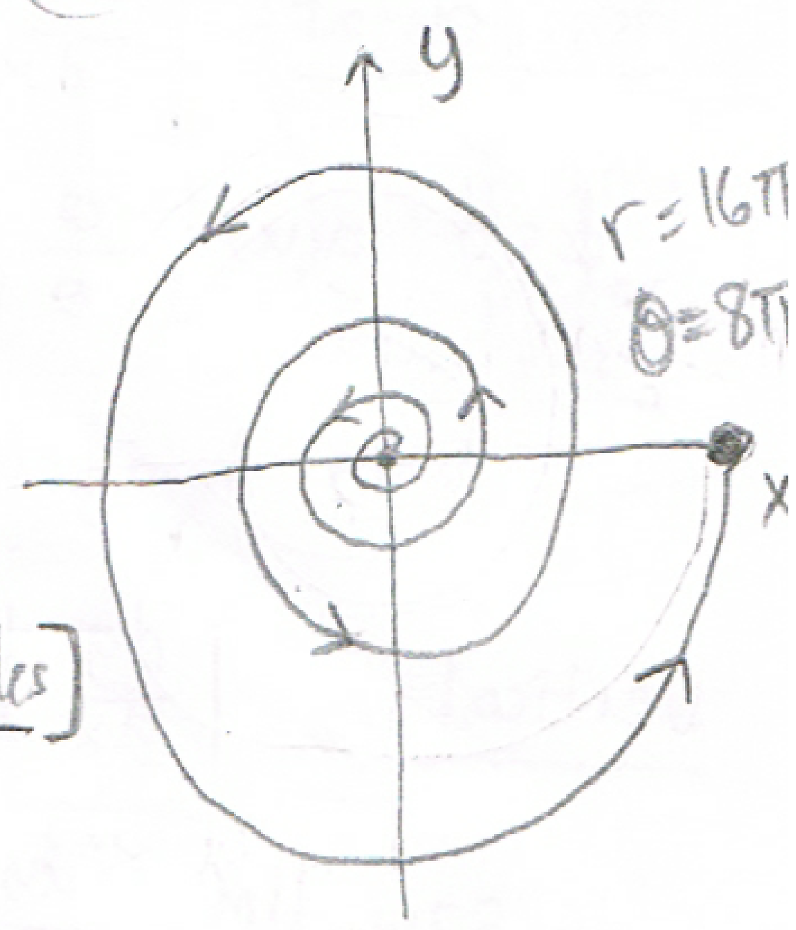


• Since $0 \leq \theta \leq 8\pi = 4 \cdot (2\pi)$, it goes around $\boxed{4 \text{ times.}}$

(sketch)

Zoom out:

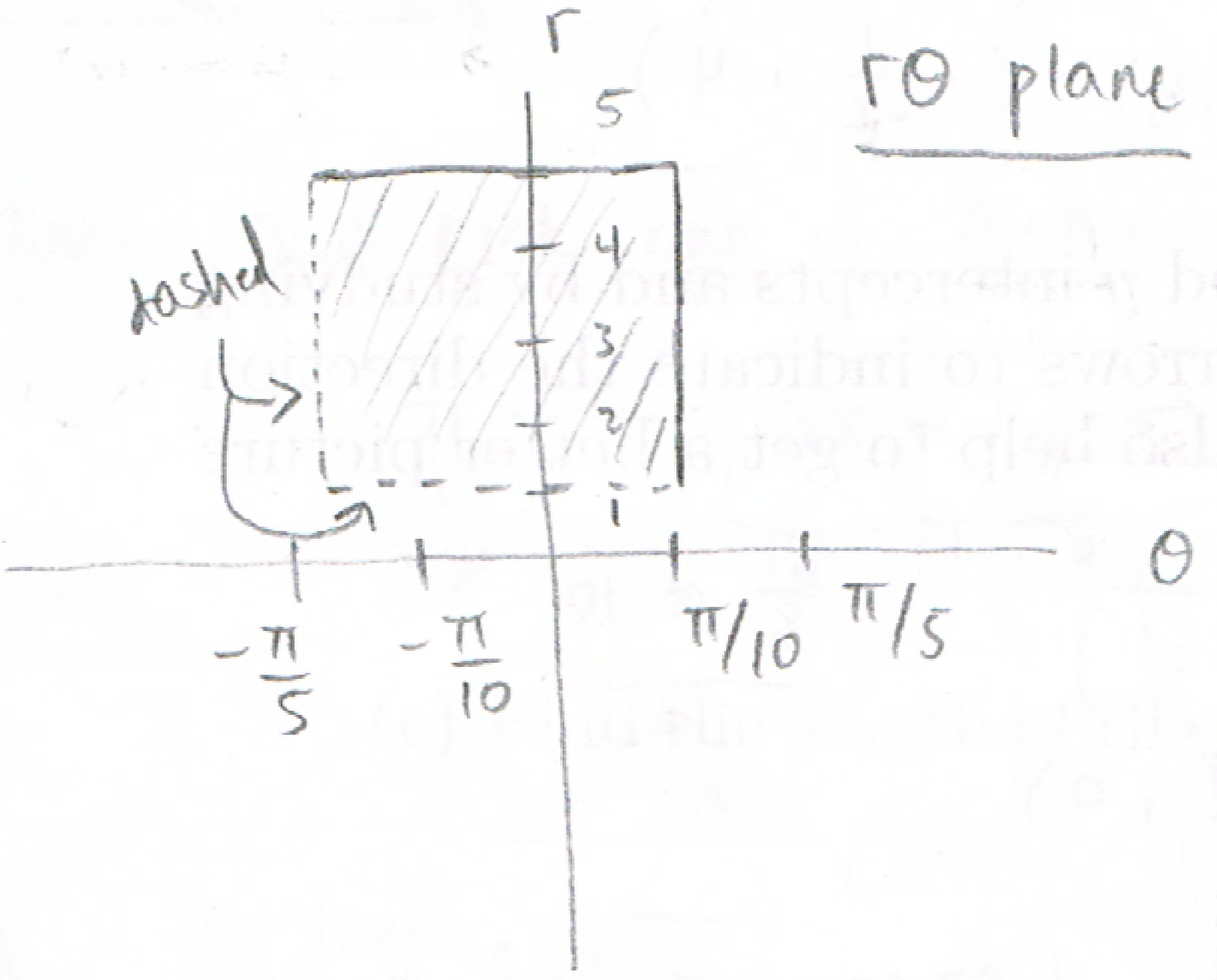
[spacing between spirals doubles]

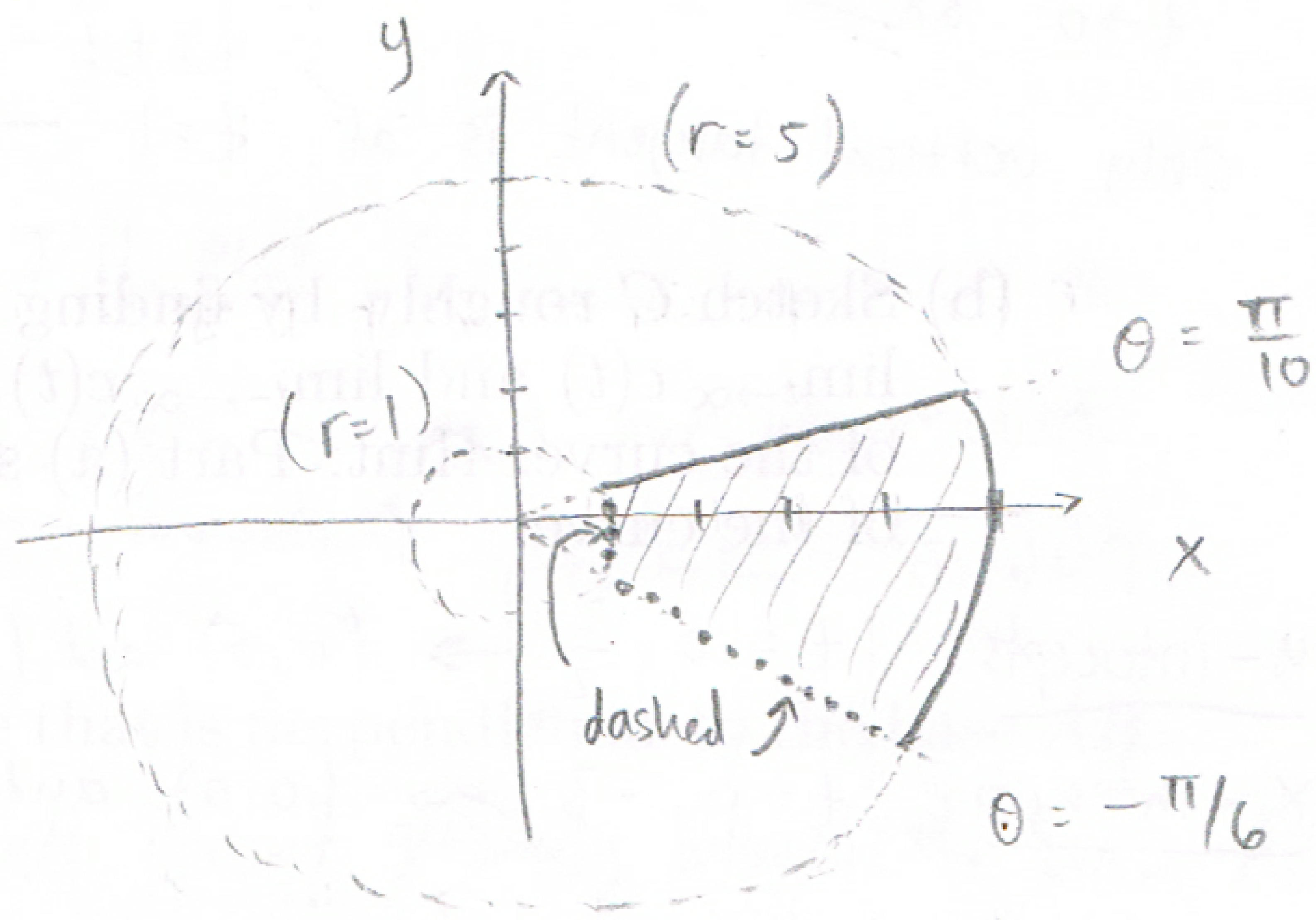


3. Sketch the region

$$1 < r \leq 5, \quad -\frac{\pi}{6} < \theta \leq \frac{\pi}{10},$$

in the xy-plane and the $r\theta$-plane.





Check (b): $d = \frac{|1+2+1-2|}{\sqrt{1+4+1}} = \frac{2}{\sqrt{6}} = \frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3}$ ✓

$(x+2y-z-2=0)$

4. Let A be the point $(1,1,-1)$, and P be the plane $x+2y-z=2$.

(a) Show that A is not on the plane P. Then find the parametric equation of the line that goes through A and is perpendicular to P.

Not on plane: Plug in $\rightsquigarrow$ $1+2(1)-(-1)=4$ but $4 \neq 2$ so it's not on P ✓

Eqn of line: Just need direction. Since perpendicular to P, direction $= \vec{n} = \langle 1,2,-1\rangle$.

Thus, $\vec{r}(t) = \langle 1,1,-1\rangle + t\langle 1,2,-1\rangle$; i.e. $\boxed{x=1+t;\ y=1+2t;\ z=-1-t}$

(b) Find the distance between the point A and the plane P.

[There's a distance formula, $d = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$] [a, b, c, d from plane equation; and $(x_1,y_1,z_1) \stackrel{!}{=} (1,1,-1)$]
(from scalar projection)

Solution: Qualitatively, dist(A, P) is the length of the (normal) line segment from A to P. In other words, we use the line in (a), since it is perpendicular to P and goes through A and P. We need when it hits P.

When line hits P: Then $x+2y-z=2 \rightsquigarrow (1+t)+2(1+2t)-(-1-t)=2$. (Plug into plane eqn to check)

Reducing, $\cancel{2}(1+t)+\cancel{2}(1+2t)=\cancel{2}$, $2+3t=1$; $t=-1/3 \rightsquigarrow$ $\boxed{x=2/3,\ y=1/3,\ z=-2/3}$

call this point Q

Thus, $\text{dist}(A,P) = |\overrightarrow{AQ}| = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{1}{9}} = \boxed{\frac{\sqrt{6}}{3}}$

(c) Find the equation of the plane that contains A and is parallel to P.

Parallel to P $\Rightarrow$ same $\vec{n} = \langle 1,2,-1\rangle$.

Hits A $\Rightarrow$ Let $(x_0,y_0,z_0) = A = (1,1,-1)$

Thus, $\boxed{1(x-1)+2(y-1)-1(z+1)=0}$ is our eqn,

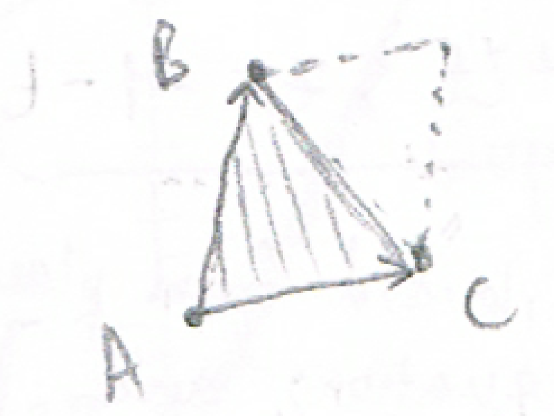
If you expand it, it's $x+2y-z=4$.

Check (a) and (c): $\frac{L \cdot |\overrightarrow{AB}|}{2} = \text{Area}_T$ ✓

5. Let $A = (1,1,1)$, $B = (2,2,0)$, $C = (0,2,3)$, and consider the triangle $T = ABC$.

(a) Find the area of T.

Note: Geometrically, it's this setup:



(For Example.)

Mainly, T is half of that parallelogram. $\left(\text{So, Area}_T = \frac{\text{Area}_{para}}{2}\right)$

We know how to get the area of parallelograms,

let $\vec{v_1} = \overrightarrow{AB} = \langle 1,1,-1 \rangle$; $\vec{v_2} = \overrightarrow{AC} = \langle -1,1,2 \rangle$.

$\text{Area}_{para} = |\vec{v_1} \times \vec{v_2}|$ then. $\vec{v_1} \times \vec{v_2} = (2+1)\hat{\imath} + (1-2)\hat{\jmath} + (1+1)\hat{k}$

So, $\text{Area}_{para} = \sqrt{9+1+4} = \sqrt{14}$

$= \det\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ -1 & 1 & 2 \end{pmatrix}$ if you prefer det.

Thus, $\boxed{\text{Area}_T = \frac{\sqrt{14}}{2}}$

(b) Find the equation of the plane that contains the triangle.

Need $\vec{n}$: cross any two vectors → The same $\vec{v_1}$ and $\vec{v_2}$ are fine.

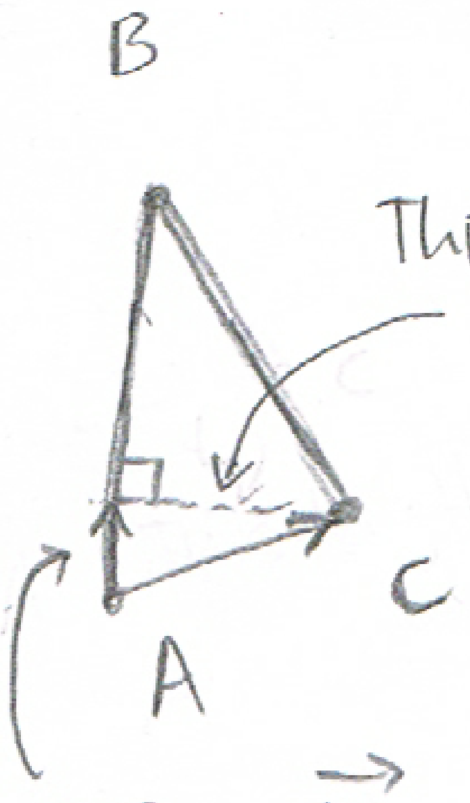
Thus, $\vec{n} = \vec{v_1} \times \vec{v_2} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$.

Point: Just pick one of A, B, or C. I'll pick C.

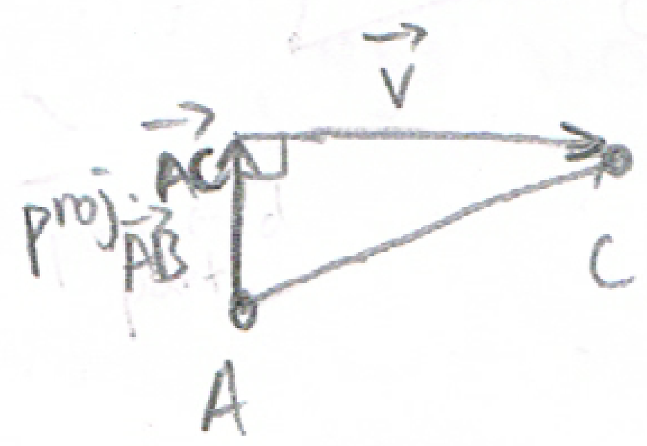
Then, $\boxed{3x - (y-2) + 2(z-3) = 0}$ is our plane which contains Triangle T.

(c) Find the length of the altitude that is perpendicular to the base AB.

⚠ Careful, the altitude doesn't generally bisect the base AB.



★ Note: $\overrightarrow{AC} = \text{proj}_{\overrightarrow{AB}} \overrightarrow{AC} + \vec{v}$ when we seek $|\vec{v}|$



$\overrightarrow{AB} = \langle 1,1,-1 \rangle$ and $\overrightarrow{AC} = \langle -1,1,2 \rangle$.

So, $\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC} = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}|^2} \overrightarrow{AB} = \frac{(-1+1-2)}{3} \langle 1,1,-1 \rangle$

Thus, $\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC} = \langle -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3} \rangle$.

So, $\vec{v} = \overrightarrow{AC} - \text{proj}_{\overrightarrow{AB}} \overrightarrow{AC} = \langle -1,1,2 \rangle - \langle -2/3, -2/3, 2/3 \rangle = \langle -\frac{1}{3}, \frac{5}{3}, \frac{4}{3} \rangle$

Thus, $L = |\vec{v}| = \sqrt{\frac{1}{9} + \frac{25}{9} + \frac{16}{9}} = \boxed{\frac{\sqrt{42}}{3} = L}$